

# Asymptotic behaviors of bivariate Gaussian powered extremes

Wei Zhou and Zuoxiang Peng\*

School of Mathematics and Statistics, Southwest University, Chongqing, 400715, China

**Abstract** In this paper, joint asymptotics of powered maxima for a triangular array of bivariate powered Gaussian random vectors are considered. Under the Hüsler-Reiss condition, limiting distributions of powered maxima are derived. Furthermore, the second-order expansions of the joint distributions of powered maxima are established under the refined Hüsler-Reiss condition.

**Keywords** Hüsler-Reiss max-stable distribution · bivariate powered Gaussian maximum · second-order expansion

**AMS 2000 subject classification** Primary 62E20, 60G70; Secondary 60F15, 60F05.

## 1 Introduction

For independent and identically distributed bivariate Gaussian random vectors with constant coefficient in each vector, Sibuya (1960) showed that componentwise maxima are asymptotically independent, and Embrechts et al. (2003) proved the asymptotical independence in the upper tail. To overcome those shortcomings in its applications, Hüsler and Reiss (1989) considered the asymptotic behaviors of extremes of Gaussian triangular arrays with varying coefficients. Precisely, let  $\{(X_{ni}, Y_{ni}), 1 \leq i \leq n, n \geq 1\}$  be a triangular array of independent bivariate Gaussian random vectors with  $E X_{ni} = E Y_{ni} = 0$ ,  $\text{Var } X_{ni} = \text{Var } Y_{ni} = 1$  for  $1 \leq i \leq n, n \geq 1$ . and  $\text{Cov}(X_{ni}, Y_{ni}) = \rho_n$ . Let  $F_{\rho_n}(x, y)$  denote the joint distribution of vector  $(X_{ni}, Y_{ni})$  for  $i \leq n$ . The partial maxima

---

\*Corresponding author. Email: pzx@swu.edu.cn

$\mathbf{M}_n$  is defined by

$$\mathbf{M}_n = (M_{n1}, M_{n2}) = (\max_{1 \leq i \leq n} X_{ni}, \max_{1 \leq i \leq n} Y_{ni}).$$

Hüsler and Reiss (1989) and Kabluchko (2009) showed that

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( M_{n1} \leq b_n + \frac{x}{b_n}, M_{n2} \leq b_n + \frac{y}{b_n} \right) = H_\lambda(x, y) \quad (1.1)$$

holds if and only if the following Hüsler-Reiss condition

$$\lim_{n \rightarrow \infty} b_n^2(1 - \rho_n) = 2\lambda^2 \in [0, \infty] \quad (1.2)$$

holds, where the normalizing constant  $b_n$  satisfying

$$1 - \Phi(b_n) = n^{-1} \quad (1.3)$$

and the max-stable Hüsler-Reiss distribution is given by

$$H_\lambda(x, y) = \exp \left( -\Phi \left( \lambda + \frac{x-y}{2\lambda} \right) e^{-y} - \Phi \left( \lambda + \frac{y-x}{2\lambda} \right) e^{-x} \right), \quad x, y \in \mathbb{R}, \quad (1.4)$$

where  $\Phi(\cdot)$  and  $\varphi(\cdot)$  denote respectively the distribution function and density function of a standard Gaussian random variable. Note that  $H_0(x, y) = \Lambda(\min(x, y))$  and  $H_\infty(x, y) = \Lambda(x)\Lambda(y)$  with  $\Lambda(x) = \exp(-\exp(-x))$ ,  $x \in \mathbb{R}$ .

Recently, contributions to Hüsler-Reiss distribution and its extensions are achieved considerably. For instance, Hashorva (2005, 2006) showed that the limit distributions of maxima also holds for triangular arrays of general bivariate elliptical distributions if the distribution of random radius is in the Gumbel or Weibull max-domain of attraction, and Hashorva and Ling (2016) extended the results to bivariate skew elliptical triangular arrays. For more work on asymptotics of bivariate triangular arrays, see Hashorva (2008, 2013) and Hashorva et al. (2012).

Higher-order expansions of distributions of extremes on Hüsler-Reiss bivariate Gaussian triangular arrays were considered firstly by Hashorva et al. (2016) provided that  $\rho_n$  satisfies the following

refined Hüsler-Reiss condition

$$\lim_{n \rightarrow \infty} b_n^2(\lambda_n - \lambda) = \alpha \in \mathbb{R}, \quad (1.5)$$

where  $\lambda_n = (\frac{1}{2}b_n^2(1 - \rho_n))^{1/2}$  and  $\lambda \in (0, \infty)$ , with  $b_n$  given by (1.3). Uniform convergence rate was considered by Liao and Peng (2014). For copula version of the limit in Hüsler-Reiss model, Frick and Reiss (2013) considered the penultimate and ultimate convergence rates for distribution of  $(n(\max_{1 \leq i \leq n} \Phi(X_{ni}) - 1), n((\max_{1 \leq i \leq n} \Phi(Y_{ni}) - 1)))$ , and Liao et al. (2016) extended the results to the settings of  $n$  independent and non-identically distributed observations, where the  $i$ th observation follows from normal copula with correlation coefficient being either a parametric or a nonparametric function of  $i/n$ .

The objective of this paper is to study the asymptotics of powered-extremes of Hüsler-Reiss bivariate Gaussian triangular arrays. Interesting results in Hall (1980) showed that the convergence rates of the distributions of powered-extremes of independent and identically distributed univariate Gaussian sequence depend on the power index and normalizing constants. Precisely, Let  $|M_n|^t$  denote the powered maximum with any power index  $t > 0$ , then

$$\lim_{n \rightarrow \infty} b_n^2 \left[ \mathbb{P}(|M_n|^t \leq c_n x + d_n) - \Lambda(x) \right] = \Lambda(x) \mu(x) \quad (1.6)$$

with normalizing constants  $c_n$  and  $d_n$  given by

$$c_n = t b_n^{t-2}, \quad d_n = b_n^t, \quad t > 0. \quad (1.7)$$

Furthermore, for  $t = 2$  with normalizing constants  $c_n^*$  and  $d_n^*$  given by

$$c_n^* = 2 - 2b_n^{-2}, \quad d_n^* = b_n^2 - 2b_n^{-2}, \quad (1.8)$$

we have

$$\lim_{n \rightarrow \infty} b_n^4 \left[ \mathbb{P}(|M_n|^2 \leq c_n^* x + d_n^*) - \Lambda(x) \right] = \Lambda(x) \nu(x), \quad (1.9)$$

where  $b_n$  is defined in (1.3), and  $\mu(x)$  and  $\nu(x)$  are respectively given by

$$\mu(x) = \left(1 + x + \frac{2-t}{2}x^2\right)e^{-x}, \quad \nu(x) = -\left(\frac{7}{2} + 3x + x^2\right)e^{-x}. \quad (1.10)$$

Motivated by findings of Hüsler-Reiss (1989), Hall (1980) and Hashorva et al. (2016), we will consider the distributional asymptotics of powered-extremes of Hüsler-Reiss bivariate Gaussian triangular arrays, and hope that the convergence rates can be improved as  $t = 2$ , similar to (1.9) in univariate case. Unfortunately, our results provide negative answers except two extreme cases.

The rest of the paper is organized as follows. In Section 2 we provide the main results and all proofs are deferred to Section 4. Some auxiliary results are given in Section 3.

## 2 Main Results

In this section, the limiting distributions and the second-order expansions on distributions of normalized bivariate powered-extremes are provided if  $\rho_n$  satisfies (1.2) and (1.5), respectively. The first main result, stated as follows, is the limit distributions of bivariate normalized powered-extremes.

**Theorem 2.1.** *Let the norming constants  $c_n$  and  $d_n$  be given by (1.7). Assume that (1.2) holds with  $\lambda \in (0, \infty)$ . Then for all  $x, y \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(|M_{n1}|^t \leq c_n x + d_n, |M_{n2}|^t \leq c_n y + d_n) = H_\lambda(x, y). \quad (2.1)$$

**Remark 2.1.** *For  $t = 2$ , with arguments similar to the proof of Theorem 2.1 one can show that (2.1) also holds with  $c_n$  and  $d_n$  being replaced by  $c_n^*$  and  $d_n^*$  given by (1.8).*

Next we investigate the convergence rate of

$$\Delta(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) = \mathbb{P}(|M_{n1}|^t \leq c_n x + d_n, |M_{n2}|^t \leq c_n y + d_n) - H_\lambda(x, y) \rightarrow 0 \quad (2.2)$$

as  $n \rightarrow \infty$  under the refined second-order Hüsler-Reiss condition (1.5). The results are stated as follows.

**Theorem 2.2.** *If the second Hüsler-Reiss condition (1.5) holds with  $\lambda_n = (\frac{1}{2}b_n^2(1 - \rho_n))^{1/2}$  and  $\lambda \in (0, \infty)$ , then for all  $x, y \in \mathbb{R}$ , we have*

$$\lim_{n \rightarrow \infty} (\log n) \Delta(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) = \frac{1}{2} \tau(\alpha, \lambda, x, y, t) H_\lambda(x, y) \quad (2.3)$$

with

$$\begin{aligned} \tau(\alpha, \lambda, x, y, t) = & \mu(x) \Phi\left(\lambda + \frac{y-x}{2\lambda}\right) + \mu(y) \Phi\left(\lambda + \frac{x-y}{2\lambda}\right) \\ & + \left(2\alpha - (x+y+2)\lambda - \lambda^3\right) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right), \end{aligned}$$

where  $\mu(x)$  is the one given by (1.10).

**Remark 2.2.** *For  $t = 2$ , let  $c_n$  and  $d_n$  be replaced by  $c_n^*$  and  $d_n^*$  respectively in (1.8), one can show that*

$$\lim_{n \rightarrow \infty} (\log n) \Delta(F_{\rho_n, 2}^n, H_\lambda; c_n^*, d_n^*; x, y) = \frac{1}{2} \chi(\alpha, \lambda, x, y) H_\lambda(x, y)$$

with

$$\chi(\alpha, \lambda, x, y) = \left(2\alpha - (x+y+2)\lambda - \lambda^3\right) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda}\right).$$

*The result shows the fact that the convergence rates can not be improved as  $t = 2$  with normalizing constants  $c_n^*$  and  $d_n^*$ , contrary to the result of univariate Gaussian case provided by Hall (1980).*

In order to obtain the convergence rates of (2.2) for two extreme cases  $\lambda = 0$  and  $\lambda = \infty$ , we may need some additional conditions. Following results show that rates of convergence are considerably different with different choice of normalizing constants. With power index  $t > 0$  and normalizing constants  $c_n$  and  $d_n$  given by (1.7), the results are stated as follows.

**Theorem 2.3.** *Let  $c_n$  and  $d_n$  be given by (1.7). With  $x, y \in \mathbb{R}$  and  $t > 0$  we have the following results.*

(a). *For the case of  $\lambda = \infty$ ,*

(i) if  $\rho_n \in [-1, 0]$ , we have

$$\lim_{n \rightarrow \infty} (\log n) \Delta(F_{\rho_n, t}^n, H_\infty; c_n, d_n; x, y) = \frac{1}{2} (\mu(x) + \mu(y)) H_\infty(x, y). \quad (2.4)$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$ , then (2.4) also holds.

(b). For the case of  $\lambda = 0$ ,

(i) if  $\rho_n = 1$ , we have

$$\lim_{n \rightarrow \infty} (\log n) \Delta(F_{\rho_n, t}^n, H_0; c_n, d_n; x, y) = \frac{1}{2} \mu(\min(x, y)) H_0(x, y). \quad (2.5)$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} b_n^6(1 - \rho_n) = c_1 \in [0, \infty)$ , then (2.5) also holds.

Theorem 2.3 shows that convergence rates of (2.2) are the same order of  $1/\log n$  if we choose the normalizing constants  $c_n$  and  $d_n$  given by (1.7) as  $t > 0$ . With another pair of normalizing constants  $c_n^*$  and  $d_n^*$  given by (1.8), the following results show that convergence rates of (2.2) can be improved.

**Theorem 2.4.** For  $t = 2$ , let  $c_n^*$  and  $d_n^*$  be given by (1.8). With  $x, y \in \mathbb{R}$  we have the following results.

(a). For the case of  $\lambda = \infty$ ,

(i) if  $\rho_n \in [-1, 0]$ , we have

$$\lim_{n \rightarrow \infty} (\log n)^2 \Delta(F_{\rho_n, 2}^n, H_\infty; c_n^*, d_n^*; x, y) = \frac{1}{4} (\nu(x) + \nu(y)) H_\infty(x, y). \quad (2.6)$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$ , then (2.6) also holds.

(b). For the case of  $\lambda = 0$ ,

(i) if  $\rho_n = 1$ , we have

$$\lim_{n \rightarrow \infty} (\log n)^2 \Delta(F_{\rho_n, 2}^n, H_0; c_n^*, d_n^*; x, y) = \frac{1}{4} \nu(\min(x, y)) H_0(x, y). \quad (2.7)$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} b_n^{14}(1 - \rho_n) = c_2 \in [0, \infty)$ , then (2.7) also holds.

### 3 Auxiliary Lemmas

For notational simplicity, let

$$\omega_{n,t}(x) = (c_n x + d_n)^{1/t} \quad \text{for } t > 0, \quad \text{and} \quad \omega_{n,2}^*(x) = (c_n^* x + d_n^*)^{1/2} \quad \text{as } t = 2, \quad (3.1)$$

where the normalizing constants  $c_n$  and  $d_n$ , and  $c_n^*$  and  $d_n^*$  are those given by (1.7) and (1.8), respectively. Define

$$\bar{\Phi}(z) = 1 - \Phi(z), \quad \bar{\Phi}_{n,t}(z) = n\bar{\Phi}(\omega_{n,t}(z))$$

and

$$I_k := \int_y^\infty \varphi \left( \lambda + \frac{x-z}{2\lambda} \right) e^{-z} z^k dz, \quad k = 0, 1, 2. \quad (3.2)$$

**Lemma 3.1.** *Under the conditions of Theorem 2.1, we have*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( M_{n1} \leq \omega_{n,t}(x), M_{n2} \leq \omega_{n,t}(y) \right) = H_\lambda(x, y) \quad (3.3)$$

**Proof.** With the choice of  $c_n$  and  $d_n$  in (1.7), it follows from (3.1) that

$$\omega_{n,t}(z) = (c_n z + d_n)^{1/t} = b_n \left( 1 + z b_n^{-2} + \frac{1-t}{2} z^2 b_n^{-4} + O(b_n^{-6}) \right)$$

for fixed  $z$ , hence for fixed  $x$  and  $z$ ,

$$\begin{aligned} & \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \\ = & b_n \sqrt{\frac{1 - \rho_n}{1 + \rho_n}} + \frac{x - z}{b_n \sqrt{1 - \rho_n^2}} + \frac{z}{b_n} \sqrt{\frac{1 - \rho_n}{1 + \rho_n}} + \frac{(1-t)(x^2 - z^2)}{2b_n^3 \sqrt{1 - \rho_n^2}} + \frac{(1-t)z^2}{2b_n^3} \sqrt{\frac{1 - \rho_n}{1 + \rho_n}} \\ & + \sqrt{\frac{1 - \rho_n}{1 + \rho_n}} O(b_n^{-5}) \end{aligned}$$

$$= \left( \lambda_n + \frac{x-z}{2\lambda_n} \left( 1 + \frac{(1-t)(x+z)}{2b_n^2} \right) + \frac{\lambda_n z}{b_n^2} + \frac{(1-t)\lambda_n z^2}{2b_n^4} + \lambda_n O(b_n^{-6}) \right) \left( 1 - \frac{\lambda_n^2}{b_n^2} \right)^{-\frac{1}{2}} \quad (3.4)$$

which implies that

$$\lim_{n \rightarrow \infty} \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} = \lambda + \frac{x-z}{2\lambda} \quad (3.5)$$

holds since  $\lambda_n \rightarrow \lambda$  as  $n \rightarrow \infty$ .

With  $a_n = 1/b_n$  it follows from (3.5) that

$$\begin{aligned} & n \mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) \\ &= n \int_{\omega_{n,t}(y)}^{\infty} \bar{\Phi} \left( \frac{\omega_{n,t}(x) - \rho_n z}{\sqrt{1 - \rho_n^2}} \right) d\Phi(z) \\ &= \frac{b_n}{\varphi(b_n)} (1 - b_n^{-2} + O(b_n^{-4}))^{-1} \int_y^{\infty} \bar{\Phi} \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \varphi(b_n(1 + tza_n^2)^{1/t}) d(b_n(1 + tza_n^2)^{1/t}) \\ &= (1 + b_n^{-2} + O(b_n^{-4})) \int_y^{\infty} \bar{\Phi} \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\ &\rightarrow \int_y^{\infty} \bar{\Phi} \left( \lambda + \frac{x-z}{2\lambda} \right) e^{-z} dz \\ &= e^{-y} + e^{-x} - \Phi \left( \lambda + \frac{x-y}{2\lambda} \right) e^{-y} - \Phi \left( \lambda + \frac{y-x}{2\lambda} \right) e^{-x} \end{aligned} \quad (3.6)$$

as  $n \rightarrow \infty$ . Meanwhile, one can check that

$$\lim_{n \rightarrow \infty} \bar{\Phi}_{n,t}(x) = e^{-x}. \quad (3.7)$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} & \mathbb{P}(M_{n1} \leq \omega_{n,t}(x), M_{n2} \leq \omega_{n,t}(y)) \\ &= \exp \left[ -\bar{\Phi}_{n,t}(x) - \bar{\Phi}_{n,t}(y) + n \mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) + o(1) \right] \\ &\rightarrow H_\lambda(x, y) \end{aligned}$$

as  $n \rightarrow \infty$ . The desired result follows.  $\square$

Following result is useful to the proof of Lemma 3.3.



**Lemma 3.2.** *With  $a_n = 1/b_n$ , for large  $n$  we have*

$$\begin{aligned} & \int_y^\infty \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\ &= \int_y^\infty \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \left( 1 + \left( (1-t)z - \frac{2-t}{2} z^2 \right) b_n^{-2} \right) e^{-z} dz + O(b_n^{-4}). \end{aligned} \quad (3.8)$$

**Proof.** First note for large  $n$  and  $|x| \leq \frac{b_n^2}{4(4+t)}$ ,

$$\begin{aligned} & \left| \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + txa_n^2)^{2/t} \right) \right) (1 + txa_n^2)^{1/t-1} - e^{-x} \left( 1 + \frac{1}{b_n^2} \left( (1-t)x - \frac{2-t}{2} x^2 \right) \right) \right| \\ & \leq b_n^{-4} s(x) \exp \left( -x + \frac{|x|}{4} \right), \end{aligned} \quad (3.9)$$

where  $a_n = 1/b_n$  and  $s(x) \geq 0$  is a polynomial on  $x$  independent of  $n$ , cf. Lemma 3.2 in Li and Peng (2016).

It follows from (3.9) that

$$\begin{aligned} & \int_y^{4 \log b_n} \left| \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} \right. \\ & \quad \left. - \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) e^{-z} \left( 1 + b_n^{-2} \left( (1-t)z - \frac{2-t}{2} z^2 \right) \right) \right| dz \\ & \leq \int_y^{4 \log b_n} \left| \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \right| b_n^{-4} s(z) \exp \left( -\frac{3z}{4} \right) dz \\ & < b_n^{-4} \int_y^{4 \log b_n} s(z) \exp \left( -\frac{3z}{4} \right) dz \\ & = O(b_n^{-4}) \end{aligned} \quad (3.10)$$

and

$$\begin{aligned} & \int_{4 \log b_n}^\infty e^{-\frac{z}{2}} \left( e^{-\frac{z}{2}} \left( 1 + b_n^{-2} \left( |1-t|z + \frac{|2-t|}{2} z^2 \right) \right) \right) dz \\ & \leq e^{-2 \log b_n} \left( 1 + b_n^{-2} (4|1-t| \log b_n + 8|2-t| (\log b_n)^2) \right) \int_{4 \log b_n}^\infty e^{-\frac{z}{2}} dz \\ & = 2b_n^{-4} \left( 1 + b_n^{-2} (4|1-t| \log b_n + 8|2-t| (\log b_n)^2) \right) \\ & = O(b_n^{-4}). \end{aligned} \quad (3.11)$$

So, the remainder is to show

$$A_n = \int_{4 \log b_n}^{\infty} \exp\left(\frac{b_n^2}{2} \left(1 - (1 + tza_n^2)^{2/t}\right)\right) (1 + tza_n^2)^{1/t-1} dz = O(b_n^{-4}) \quad (3.12)$$

for large  $n$ . We check (3.12) in turn for  $0 < t < 1$  and  $t \geq 1$ .

For  $0 < t < 1$ , separate  $A_n$  into the following two parts.

$$\begin{aligned} A_{n1} &= \int_{4 \log b_n}^{2(\frac{1}{t}-1)b_n^2} \exp\left(\frac{b_n^2}{2} \left(1 - (1 + tza_n^2)^{2/t}\right)\right) (1 + tza_n^2)^{1/t-1} dz \\ &< \int_{4 \log b_n}^{2(\frac{1}{t}-1)b_n^2} e^{-z} (1 + 2(1-t))^{1/t-1} dz \\ &= O(b_n^{-4}) \end{aligned} \quad (3.13)$$

since  $\exp\left(\frac{b_n^2}{2} \left(1 - (1 + tza_n^2)^{2/t}\right)\right) < e^{-z}$ . For the second part,

$$\begin{aligned} A_{n2} &= \int_{2(1/t-1)b_n^2}^{\infty} \exp\left(\frac{b_n^2}{2} \left(1 - (1 + tza_n^2)^{2/t}\right)\right) (1 + tza_n^2)^{1/t-1} dz \\ &< \int_{2(1/t-1)b_n^2}^{\infty} e^{-z} (tza_n^2)^{1/t-1} \left(1 + \frac{1}{tza_n^2}\right)^{1/t-1} dz \\ &< (ta_n^2)^{1/t-1} \left(1 + \frac{1}{2(1-t)}\right)^{1/t-1} \int_{2(1/t-1)b_n^2}^{\infty} e^{-z} z^{1/t-1} dz \\ &= 2(3-2t)^{1/t-1} e^{-2(1/t-1)b_n^2} \\ &= o(b_n^{-4}). \end{aligned} \quad (3.14)$$

Hence, (3.13) and (3.14) shows that (3.12) holds as  $0 < t < 1$ .

Now changing to the case of  $t \geq 1$ , by using Mills' inequality we have

$$\begin{aligned} A_n &= \int_{4 \log b_n}^{\infty} \exp\left(\frac{b_n^2}{2}\right) \exp\left(-\frac{b_n^2(1 + \frac{tz}{b_n^2})^{2/t}}{2}\right) (1 + \frac{tz}{b_n^2})^{1/t-1} dz \\ &= b_n \exp\left(\frac{b_n^2}{2}\right) \int_{b_n \left(1 + \frac{4t \log b_n}{b_n^2}\right)}^{\infty} \exp\left(-\frac{s^2}{2}\right) ds \\ &= \sqrt{2\pi} b_n \exp\left(\frac{b_n^2}{2}\right) \left(1 - \Phi\left(b_n \left(1 + \frac{4t \log b_n}{b_n^2}\right)^{1/t}\right)\right) \end{aligned}$$

$$\begin{aligned}
&< \frac{\exp\left(\frac{b_n^2}{2}\left(1 - \left(1 + \frac{4t \log b_n}{b_n^2}\right)^{2/t}\right)\right)}{\left(1 + \frac{4t \log b_n}{b_n^2}\right)^{1/t}} \\
&< \frac{\exp\left(-4 \log b_n + \frac{8(t-2)(\log b_n)^2}{b_n^2}\right)}{\left(1 + \frac{4t \log b_n}{b_n^2}\right)^{1/t}} \\
&= O(b_n^{-4})
\end{aligned} \tag{3.15}$$

since  $(1+s)^{\frac{2}{t}} \geq 1 + \frac{2}{t}s + \frac{1}{t}\left(\frac{2}{t}-1\right)s^2$  for  $s > 0$ .

Combining with (3.10)-(3.12), the proof of (3.8) is complete.  $\square$

In order to show the second order asymptotic expansions of extreme value distributions, let

$$\tilde{\Delta}(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) = \mathbb{P}(M_{n1} \leq \omega_{n,t}(x), M_{n2} \leq \omega_{n,t}(y)) - H_\lambda(x, y). \tag{3.16}$$

**Lemma 3.3.** *Assume that the conditions of Theorem 2.2 hold. Then,*

$$\lim_{n \rightarrow \infty} (\log n) \tilde{\Delta}(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) = \frac{1}{2} \tau(\alpha, \lambda, x, y, t) H_\lambda(x, y) \tag{3.17}$$

where  $\tau(\alpha, \lambda, x, y, t)$  is the one given in Theorem 2.2.

**Proof.** By using (3.6) and (3.8), we have

$$\begin{aligned}
&n \mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) \\
&= \bar{\Phi}_{n,t}(y) - \int_y^\infty \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}}\right) e^{-z} \left(1 + \left((1-t)z - \frac{2-t}{2}z^2\right) b_n^{-2}\right) dz + O(b_n^{-4}).
\end{aligned}$$

for large  $n$ . It follows from (3.4) and (3.5) that

$$\begin{aligned}
&b_n^2 \int_y^\infty \left(\lambda + \frac{x-z}{2\lambda} - \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}}\right) \varphi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz \\
&\rightarrow \left(\alpha - \frac{1}{2}\lambda^3 - \frac{1}{2}\alpha\lambda^{-2}x - \frac{1}{4}\lambda x - \frac{1-t}{4\lambda}x^2\right) I_0 - \left(\frac{3}{4}\lambda - \frac{1}{2}\alpha\lambda^{-2}\right) I_1 + \frac{1-t}{4\lambda} I_2 \\
&= \kappa_1(\alpha, \lambda, x, y, t)
\end{aligned} \tag{3.18}$$

as  $n \rightarrow \infty$ , where  $I_k$  is the one given by (3.2) and

$$\begin{aligned} & \kappa_1(\alpha, \lambda, x, y, t) \\ &= 2 \left( (2-t)\lambda^4 - (2-t)\lambda^2 x + (1-t)\lambda^2 \right) \bar{\Phi} \left( \lambda + \frac{y-x}{2\lambda} \right) e^{-x} \\ & \quad + \left( 2\alpha - (5-2t)\lambda^3 + (1-t)\lambda x + (1-t)\lambda y \right) \varphi \left( \lambda + \frac{y-x}{2\lambda} \right) e^{-x}. \end{aligned}$$

Note that by Taylor's expansion with Lagrange remainder term,

$$\begin{aligned} & \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} \right) \\ &= \Phi \left( \lambda + \frac{x-z}{2\lambda} \right) + \varphi \left( \lambda + \frac{x-z}{2\lambda} \right) \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda} \right) \\ & \quad + \frac{1}{2} v_n \varphi(v_n) \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda} \right)^2, \end{aligned} \quad (3.19)$$

where  $v_n$  is between  $\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}$  and  $\lambda + \frac{x-z}{2\lambda}$ . By arguments similar to (3.18), one can check that

$$\int_y^\infty \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda} \right)^2 v_n \varphi(v_n) e^{-z} dz = O(b_n^{-4}) \quad (3.20)$$

holds for large  $n$ . Hence from (3.18), (3.19) and (3.20), it follows that

$$\lim_{n \rightarrow \infty} b_n^2 \int_y^\infty \left( \Phi \left( \lambda + \frac{x-z}{2\lambda} \right) - \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1-\rho_n^2}} \right) \right) e^{-z} dz = \kappa_1(\alpha, \lambda, x, y, t). \quad (3.21)$$

Note that

$$\bar{\Phi}_{n,t}(x) = e^{-x} - b_n^{-2} \mu(x) + O(b_n^{-4}), \quad (3.22)$$

cf. Theorem 1 in Hall (1980). Now combining with (3.6), (3.21) and (3.22), we have

$$b_n^2 \left[ \mathbb{P}(M_{n1} \leq \omega_{n,t}(x), M_{n2} \leq \omega_{n,t}(y)) - H_\lambda(x, y) \right]$$

$$\begin{aligned}
&= b_n^2 H_\lambda(x, y)(1 + o(1)) \left[ -\bar{\Phi}_{n,t}(x) - \bar{\Phi}_{n,t}(y) + n \mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) \right. \\
&\quad \left. + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right) e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x} \right] \\
&= b_n^2 H_\lambda(x, y)(1 + o(1)) \left[ -\bar{\Phi}_{n,t}(x) + e^{-x} + \int_y^\infty \Phi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} dz \right. \\
&\quad \left. - \int_y^\infty \Phi\left(\frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}}\right) e^{-z} \left(1 + \left((1-t)z - \frac{2-t}{2}z^2\right) b_n^{-2}\right) dz + O(b_n^{-4}) \right] \\
&\rightarrow H_\lambda(x, y) \left[ \mu(x) + \kappa_1(\alpha, \lambda, x, y, t) - \kappa_2(\alpha, \lambda, x, y, t) \right] \\
&= H_\lambda(x, y) \tau(\alpha, \lambda, x, y, t)
\end{aligned}$$

as  $n \rightarrow \infty$ , where where

$$\begin{aligned}
&\kappa_2(\alpha, \lambda, x, y, t) \\
&= \int_y^\infty \Phi\left(\lambda + \frac{x-z}{2\lambda}\right) e^{-z} \left((1-t)z - \frac{2-t}{2}z^2\right) dz \\
&= -\left(\frac{2-t}{2}y^2 + y + 1\right) \Phi\left(\lambda + \frac{x-y}{2\lambda}\right) e^{-y} \\
&\quad + \left(2(2-t)\lambda^4 - 2(2-t)\lambda^2x + 2(1-t)\lambda^2 + \frac{2-t}{2}x^2 + x + 1\right) \bar{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x} \\
&\quad - \left(2(2-t)\lambda^3 - (2-t)\lambda(x+y) - 2\lambda\right) \varphi\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x}
\end{aligned}$$

and  $\tau(\alpha, \lambda, x, y, t)$  is the one given by Theorem 2.2. The proof is complete.  $\square$

**Lemma 3.4.** *With  $c_n$  and  $d_n$  given by (1.7), the following results hold.*

(a). *For the case of  $\lambda = \infty$ ,*

(i) *if  $\rho_n \in [-1, 0]$ , we have*

$$\lim_{n \rightarrow \infty} (\log n) \tilde{\Delta}(F_{\rho_n, t}^n, H_\infty; c_n, d_n; x, y) = \frac{1}{2} (\mu(x) + \mu(y)) H_\infty(x, y). \quad (3.23)$$

(ii) *if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$ , then (3.23) also holds.*

(b). *For the case of  $\lambda = 0$ ,*

(i) if  $\rho_n = 1$ , we have

$$\lim_{n \rightarrow \infty} (\log n) \tilde{\Delta}(F_{\rho_n, t}^n, H_0; c_n, d_n; x, y) = \frac{1}{2} \mu(\min(x, y)) H_0(x, y). \quad (3.24)$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} b_n^6(1 - \rho_n) = c_1 \in [0, \infty)$ , then (3.24) also holds.

**Proof.** For  $\lambda = \infty$ , we first consider (i), i.e., the case of  $\rho_n \in [-1, 0]$ . Note that either complete independent ( $\rho_n \equiv 0$ ) or complete negative dependent ( $\rho_n \equiv -1$ ) both imply  $\lambda = \infty$ . Thus from (3.22) it follows that both

$$\begin{aligned} & b_n^2 (-n(1 - F_{-1}(\omega_{n,t}(x), \omega_{n,t}(y))) + e^{-y} + e^{-x}) \\ = & b_n^2 (-\bar{\Phi}_{n,t}(x) + e^{-x}) + b_n^2 (-\bar{\Phi}_{n,t}(y) + e^{-y}) + nb_n^2 \mathbb{P}(\omega_{n,t}(x) < X < -\omega_{n,t}(y)) \\ \rightarrow & \mu(x) + \mu(y) \end{aligned} \quad (3.25)$$

and

$$\begin{aligned} & b_n^2 (-n(1 - F_0(\omega_{n,t}(x), \omega_{n,t}(y))) + e^{-y} + e^{-x}) \\ = & b_n^2 (-\bar{\Phi}_{n,t}(x) + e^{-x}) + b_n^2 (-\bar{\Phi}_{n,t}(y) + e^{-y}) + \frac{b_n^2}{n} \bar{\Phi}_{n,t}(x) \bar{\Phi}_{n,t}(y) \\ \rightarrow & \mu(x) + \mu(y) \end{aligned} \quad (3.26)$$

hold as  $n \rightarrow \infty$ , showing that the claimed results (3.23) hold for  $\rho_n \equiv -1$  and  $\rho_n \equiv 0$  respectively. Thus, it follows from Slepian's Lemma that (3.23) also holds for  $\rho_n \in [-1, 0]$ .

Now switch to the case of  $\rho_n \in (0, 1)$  with additional condition  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1 - \rho_n)} = 0$ , implying  $\lambda = \infty$ . For fixed  $x, z \in \mathbb{R}$ , one can check that

$$\lim_{n \rightarrow \infty} \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} = \infty. \quad (3.27)$$

Note that the condition  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1 - \rho_n)} = 0$  implies  $\lim_{n \rightarrow \infty} b_n^2(1 - \rho_n) = \infty$ . By (3.27) and Mills' inequality,

$$b_n^4 \left( 1 - \Phi \left( \frac{\omega_{n,t}(x) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \right)$$

$$\begin{aligned}
&< \frac{b_n^4 \exp\left(-\frac{(\omega_{n,t}(x)-\rho_n\omega_{n,t}(z))^2}{2(1-\rho_n^2)}\right)}{\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}} \\
&= \frac{\exp\left(-\frac{((c_n x+d_n)^{1/t}-\rho_n(c_n z+d_n)^{1/t})^2}{2(1-\rho_n^2)} + 4\log b_n\right)}{\frac{(c_n x+d_n)^{1/t}-\rho_n(c_n z+d_n)^{1/t}}{\sqrt{1-\rho_n^2}}} \\
&= \left(b_n \sqrt{\frac{1-\rho_n}{1+\rho_n}} + \frac{x-z}{b_n \sqrt{1-\rho_n^2}} + \frac{z}{b_n} \sqrt{\frac{1-\rho_n}{1+\rho_n}} + \frac{(1-t)(x^2-\rho_n z^2)}{2b_n^3 \sqrt{1-\rho_n^2}} + \sqrt{\frac{1-\rho_n}{1+\rho_n}} O(b_n^{-5})\right)^{-1} \\
&\quad \times \exp\left(-\frac{b_n^2(1-\rho_n)}{2(1+\rho_n)} - \frac{(x-\rho_n z)^2}{2b_n^2(1-\rho_n^2)} - \frac{(1-t)^2(x^2-\rho_n z^2)^2}{8b_n^6(1-\rho_n^2)} - \frac{x-\rho_n z}{1+\rho_n} - \frac{(1-t)(x^2-\rho_n z^2)}{2b_n^2(1+\rho_n)}\right. \\
&\quad \left.- \frac{(x-\rho_n z)(1-t)(x^2-\rho_n z^2)}{2b_n^4(1-\rho_n^2)} + \frac{1-\rho_n}{2(1+\rho_n)} O(b_n^{-5}) + 4\log b_n\right) \\
&< (1+o(1))e^{-\frac{x-z}{2}} \exp\left\{-\frac{b_n^2(1-\rho_n)}{2(1+\rho_n)} \left(1 - \frac{8(1+\rho_n)\log b_n}{b_n^2(1-\rho_n)} + \frac{(1+\rho_n)\log b_n^2(1-\rho_n)}{b_n^2(1-\rho_n)}\right)\right\} \\
&\rightarrow 0
\end{aligned} \tag{3.28}$$

as  $n \rightarrow \infty$ . Note that

$$n^{-1} = \bar{\Phi}(b_n) = \frac{\varphi(b_n)}{b_n}(1 - b_n^{-2} + O(b_n^{-4})). \tag{3.29}$$

Hence, by using (3.10)-(3.12), we have

$$\begin{aligned}
&n \mathbb{P}(X > \omega_{n,t}(x), Y > \omega_{n,t}(y)) \\
&= b_n^{-4}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \int_y^\infty b_n^4 \bar{\Phi}\left(\frac{\omega_{n,t}(x)-\rho_n\omega_{n,t}(z)}{\sqrt{1-\rho_n^2}}\right) \exp\left(\frac{b_n^2}{2} \left(1 - (1 + tza_n^2)^{2/t}\right)\right) \\
&\quad \times (1 + tza_n^2)^{1/t-1} dz \\
&= O(b_n^{-4}).
\end{aligned} \tag{3.30}$$

It follows from (3.22) and (3.30) that

$$\begin{aligned}
&b_n^2 \left[ F_{\rho_n}^n(\omega_{n,t}(x), \omega_{n,t}(y)) - H_\infty(x, y) \right] \\
&= b_n^2 H_\infty(x, y)(1 + o(1)) \left[ -n(1 - F_{\rho_n}(\omega_{n,t}(x), \omega_{n,t}(y))) + e^{-x} + e^{-y} \right] \\
&= b_n^2 H_\infty(x, y)(1 + o(1)) \left[ -(e^{-x} + e^{-y} - b_n^{-2}(\mu(x) + \mu(y) + O(b_n^{-2}))) + e^{-x} + e^{-y} \right] \\
&\rightarrow H_\infty(x, y) [\mu(x) + \mu(y)]
\end{aligned}$$

as  $n \rightarrow \infty$ . Proof the case of  $\lambda = \infty$  is complete.

(b). For the case of  $\lambda = 0$ , we first consider the complete positive dependence case ( $\rho_n \equiv 1$ ).

Without loss of generality, assume that  $y < x$ , we have

$$b_n^2 \left[ -n(1 - F_1(\omega_{n,t}(x), \omega_{n,t}(y))) + e^{-y} \right] = b_n^2 \left[ -\bar{\Phi}_{n,t}(y) + e^{-y} \right] \rightarrow \mu(y) \quad (3.31)$$

as  $n \rightarrow \infty$  since (3.22) holds. The rest is for the case of  $\rho_n \in (0, 1)$ . For  $y < x \in \mathbb{R}$ , if  $\max(x, y) = x < z < 4 \log b_n$  we have

$$\begin{aligned} & \Phi \left( \frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \\ & < - \frac{\varphi \left( \frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right)}{\frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}}} \\ & = \frac{\exp \left( -\frac{1}{2} \left( \lambda_n + \frac{y-z}{2\lambda_n} \right)^2 (1 + o(1)) \right)}{\left( -\lambda_n + \frac{z-y}{2\lambda_n} \left( 1 + \frac{(1-t)(y+z)}{2b_n^2} \right) - \frac{\lambda_n z}{b_n^2} - \frac{(1-t)\lambda_n z^2}{2b_n^4} + \lambda_n O(b_n^{-6}) \right) \left( 1 - \frac{\lambda_n^2}{b_n^2} \right)^{-\frac{1}{2}}} \\ & = \frac{\exp \left( -\frac{1}{2} \left( \lambda_n + \frac{y-z}{2\lambda_n} \right)^2 (1 + o(1)) \right)}{\frac{z-y}{2\lambda_n} (1 + o(1))} \end{aligned} \quad (3.32)$$

for large  $n$  due to  $\Phi(-x) = \bar{\Phi}(x)$  and Mills' inequality since  $\frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} < 0$  for large  $n$  when  $\lim_{n \rightarrow \infty} b_n^6(1 - \rho_n) = c_1 \in [0, \infty)$ . Therefore,

$$\begin{aligned} & \int_x^{4 \log b_n} \Phi \left( \frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\ & < \frac{2\lambda_n}{x-y} (1 + o(1)) \int_x^{4 \log b_n} \exp \left( -\frac{\lambda_n^2}{2} - \frac{y-z}{2} - \frac{(y-z)^2}{8\lambda_n^2} - z + o(b_n^{-1}) + \left( \frac{1}{t} - 1 \right) \log(1 + tza_n^2) \right) dz \\ & = 2\lambda_n (1 + o(1)) \frac{\exp \left( -\frac{\lambda_n^2}{2} - \frac{y}{2} - \frac{(y-x)^2}{8\lambda_n^2} \right)}{x-y} \int_x^{4 \log b_n} \exp \left( -\frac{z}{2} \right) dz \\ & < 4\lambda_n b_n^{-2} (1 + o(1)) \frac{\exp \left( -\frac{\lambda_n^2}{2} - \frac{y}{2} - \frac{(y-x)^2}{8\lambda_n^2} \right)}{y-x} \\ & = O(b_n^{-4}) \end{aligned} \quad (3.33)$$



for large  $n$  by using  $\lim_{n \rightarrow \infty} b_n^6(1 - \rho_n) = c_1$ . It follows from (3.12) that

$$\begin{aligned}
& \int_{4 \log b_n}^{\infty} \Phi \left( \frac{(\omega_{n,t}(y) - \rho_n \omega_{n,t}(z))^2}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\
& < \int_{4 \log b_n}^{\infty} \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\
& = O(b_n^{-4}).
\end{aligned} \tag{3.34}$$

Combining (3.22), (3.33) and (3.34), for  $y < x$  we have

$$\begin{aligned}
& 1 - F_{\rho_n}(\omega_{n,t}(\min(x, y)), \omega_{n,t}(\max(x, y))) \\
& = \bar{\Phi}_{n,t}(y) + \bar{\Phi}_{n,t}(x) - \mathbb{P}(X > \omega_{n,t}(y), Y > \omega_{n,t}(x)) \\
& = \bar{\Phi}_{n,t}(y) + \bar{\Phi}_{n,t}(x) \\
& \quad - \int_x^{\infty} \bar{\Phi} \left( \frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\
& = \bar{\Phi}_{n,t}(y) + n^{-1}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \\
& \quad \times \int_x^{\infty} \Phi \left( \frac{\omega_{n,t}(y) - \rho_n \omega_{n,t}(z)}{\sqrt{1 - \rho_n^2}} \right) \exp \left( \frac{b_n^2}{2} \left( 1 - (1 + tza_n^2)^{2/t} \right) \right) (1 + tza_n^2)^{1/t-1} dz \\
& = n^{-1}(e^{-y} - b_n^{-2}\mu(y) + O(b_n^{-4}))
\end{aligned}$$

holds for large  $n$ , which implies the desired result. The proof is complete.  $\square$

**Lemma 3.5.** *For  $t = 2$ , with  $c_n^*$  and  $d_n^*$  given by (1.8), the following results hold.*

(a). *For the case of  $\lambda = \infty$ ,*

(i) *if  $\rho_n \in [-1, 0]$ , we have*

$$\lim_{n \rightarrow \infty} (\log n)^2 \tilde{\Delta}(F_{\rho_n, 2}^n, H_{\infty}; c_n^*, d_n^*; x, y) = \frac{1}{4} (\nu(x) + \nu(y)) H_{\infty}(x, y). \tag{3.35}$$

(ii) *if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1 - \rho_n)} = 0$ , then (3.35) also holds.*

(b). *For the case of  $\lambda = 0$ ,*

(i) *if  $\rho_n = 1$ , we have*

$$\lim_{n \rightarrow \infty} (\log n)^2 \tilde{\Delta}(F_{\rho_n, 2}^n, H_0; c_n^*, d_n^*; x, y) = \frac{1}{4} \nu(\min(x, y)) H_0(x, y). \tag{3.36}$$

(ii) if  $\rho_n \in (0, 1)$  and  $\lim_{n \rightarrow \infty} b_n^{14}(1 - \rho_n) = c_2 \in [0, \infty)$ , then (3.36) also holds.

**Proof.** (a). For  $\lambda = \infty$ . Firstly note that

$$\bar{\Phi}_{n,2}(x) = e^{-x} - b_n^{-4}\nu(x) + O(b_n^{-6}) \quad (3.37)$$

derived by Theorem 1 in Hall (1980). Arguments similar to that of (3.25) and (3.26), by using (3.37) we have

$$b_n^4 \left( -n(1 - F_{-1}(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} \right) \rightarrow \nu(x) + \nu(y)$$

as  $n \rightarrow \infty$  for  $\rho_n \equiv -1$ , and

$$b_n^4 \left( -n(1 - F_0(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} \right) \rightarrow \nu(x) + \nu(y)$$

also holds as  $n \rightarrow \infty$  for  $\rho_n \equiv 0$ . Therefore, (3.35) holds for  $\rho_n \equiv -1$  and  $\rho_n \equiv 0$  respectively. By using Slepian's Lemma, (3.35) also holds for  $\rho_n \in [-1, 0]$ .

Now switch to the case of  $\rho_n \in (0, 1)$  with additional condition  $\lim_{n \rightarrow \infty} \frac{\log b_n}{b_n^2(1-\rho_n)} = 0$ , implying  $\lambda = \infty$ . For fixed  $x$  and  $z$ , note that

$$\frac{\omega_{n,2}^*(x) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1 - \rho_n^2}} \rightarrow \infty$$

as  $n \rightarrow \infty$  since  $\lambda_n^2 = \frac{b_n^2}{2}(1 - \rho_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . Therefore for  $t = 2$ , arguments similar to (3.28), we have

$$b_n^6 \left( 1 - \Phi \left( \frac{\omega_{n,2}^*(x) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1 - \rho_n^2}} \right) \right) \rightarrow 0$$

as  $n \rightarrow \infty$ . Hence, it follows from (3.29) that

$$\begin{aligned} & \mathbb{P}(X > \omega_{n,2}^*(x), Y > \omega_{n,2}^*(y)) \\ &= n^{-1} b_n^{-6} (1 + b_n^{-2} + O(b_n^{-4})) \int_y^\infty b_n^6 \left( 1 - \Phi \left( \frac{\omega_{n,2}(x) - \rho_n \omega_{n,2}(y)}{\sqrt{1 - \rho_n^2}} \right) \right) \end{aligned}$$

$$\begin{aligned}
& \times \exp(-z + (1+z)a_n^2) (1-a_n^2) (1+2(z-(1+z)a_n^2)a_n^2)^{-\frac{1}{2}} dz \\
& = O(n^{-1}b_n^{-6})
\end{aligned}$$

for large  $n$ . Hence,

$$b_n^4 [F_{\rho_n}^n(\omega_{n,t}^*(x), \omega_{n,t}^*(y)) - H_\infty(x, y)] \rightarrow H_\infty(x, y) [\nu(x) + \nu(y)]$$

holds as  $n \rightarrow \infty$ . The proof the case of  $\lambda = \infty$  is complete.

(b). For the case of  $\lambda = 0$ . We first consider the complete positive dependence case ( $\rho_n \equiv 1$ ), without loss of generality, assume that  $y < x$ . Hence,

$$b_n^4 \left[ -n(1 - F_1(\omega_{n,2}^*(x), \omega_{n,2}^*(y))) + \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} \right] \rightarrow \nu(y)$$

as  $n \rightarrow \infty$  provided that (3.37) holds. The remainder is to prove the case of  $\rho_n \in (0, 1)$  with  $\lim_{n \rightarrow \infty} b_n^{14}(1 - \rho_n) = c_2 \in [0, \infty)$ . By arguments similar to that of (??) and (??), for fixed  $y < x \in \mathbb{R}$  we have

$$\int_x^\infty \Phi \left( \frac{\omega_{n,2}^*(y) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1 - \rho_n^2}} \right) \exp(-z + (1+z)a_n^2) \frac{1 - a_n^2}{(1 + 2(z - (1+z)a_n^2)a_n^2)^{\frac{1}{2}}} dz = O(b_n^{-6}) \quad (3.38)$$

for large  $n$  by using  $\lim_{n \rightarrow \infty} b_n^{14}(1 - \rho_n) = c_2$ . Combining (3.37) with (3.38), we have

$$\begin{aligned}
& 1 - F_{\rho_n}(\omega_{n,2}^*(x), \omega_{n,2}^*(y)) \\
& = \bar{\Phi}_{n,2}(y) + n^{-1}(1 - b_n^{-2} + O(b_n^{-4}))^{-1} \\
& \quad \times \int_x^\infty \Phi \left( \frac{\omega_{n,2}^*(y) - \rho_n \omega_{n,2}^*(z)}{\sqrt{1 - \rho_n^2}} \right) \exp(-z + (1+z)a_n^2) \frac{1 - a_n^2}{(1 + 2(z - (1+z)a_n^2)a_n^2)^{\frac{1}{2}}} dz \\
& = n^{-1} (e^{-y} - b_n^{-4}\nu(y) + O(b_n^{-6}))
\end{aligned}$$

for large  $n$ , which implies the desired result. The proof is complete.  $\square$

## 4 Proofs

**Proof of Theorem 2.1.** Obviously,

$$\begin{aligned} & \mathbb{P}(|M_{n1}|^t \leq c_n x + d_n, |M_{n2}|^t \leq c_n y + d_n) \\ = & F_{\rho_n, t}^n(\omega_{n, t}(x), \omega_{n, t}(y)) - F_{\rho_n, t}^n(\omega_{n, t}(x), -\omega_{n, t}(y)) - F_{\rho_n, t}^n(-\omega_{n, t}(x), \omega_{n, t}(y)) + F_{\rho_n, t}^n(-\omega_{n, t}(x), -\omega_{n, t}(y)). \end{aligned}$$

Note that

$$\begin{aligned} & F_{\rho_n, t}^n(\omega_{n, t}(x), -\omega_{n, t}(y)) + F_{\rho_n, t}^n(-\omega_{n, t}(x), \omega_{n, t}(y)) - F_{\rho_n, t}^n(-\omega_{n, t}(x), -\omega_{n, t}(y)) \\ \leq & \mathbb{P}(M_{n2} \leq -\omega_{n, t}(y)) + \mathbb{P}(M_{n1} \leq -\omega_{n, t}(x)) - \min\{\Phi^n(-\omega_{n, t}(x)), \Phi^n(-\omega_{n, t}(y))\} \\ = & \Phi^n(-\omega_{n, t}(x)) + \Phi^n(-\omega_{n, t}(y)) - \min\{\Phi^n(-\omega_{n, t}(x)), \Phi^n(-\omega_{n, t}(y))\} \\ = & o(b_n^{-4}) \end{aligned} \tag{4.1}$$

since

$$\bar{\Phi}^{n-1}(-\omega_{n, t}(x)) = (n^{-1}e^{-x}(1 + O(b_n^{-2})))^{n-1} = o(b_n^{-4}),$$

cf. Lemma 3.1 in Zhou and Ling (2016). Combining (4.1) with Lemma 3.1, we can get the claimed result (2.1).  $\square$

**Proof of Theorem 2.2.** It follows from (4.1) and Lemma 3.3 that

$$\Delta(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) = \tilde{\Delta}(F_{\rho_n, t}^n, H_\lambda; c_n, d_n; x, y) + o(b_n^{-4}),$$

so the result (2.3) is obtained.  $\square$

**Proof of Theorem 2.3 and Theorem 2.4.** By using (4.1), Lemma 3.4 and Lemma 3.5 respectively to derive the desired results.  $\square$

**Acknowledgments** The first author was supported by the Fundamental Research Funds for the Central Universities Grant no. XDJK2016E117, the CQ Innovation Project for Graduates Grant no. CYS16046.

## References

- [1] Embrechts, P., Lindskog, F. and McNeil, A. (2003). Modelling dependence with copulas and applications to risk management. In *Handbook of Heavy Tailed Distributions in Finance* (ed. S. Rachev), 329–384. Elsevier, Amsterdam.
- [2] Frick, M. and Reiss, R.-D. (2010). Limiting distributions of maxima under triangular schemes. *Journal of Multivariate Analysis*, **101**, 2346–2357.
- [3] Frick, M. and Reiss, R.-D. (2013). Expansions and penultimate distributions of maxima of bivariate Gaussian random vectors. *Statistics and Probability Letters*, **83**, 2563–2568.
- [4] Hall, P. (1980). Estimating probabilities for normal extremes. *Advances in Applied Probability*, **12**, 491–500.
- [5] Hashorva, E. (2005). Elliptical triangular arrays in the max-domain of attraction of Hüsler-Reiss distribution. *Statistics and Probability Letters*, **72**, 125–135.
- [6] Hashorva, E. (2006). On the max-domain of attractions of bivariate elliptical arrays. *Extremes*, **8**, 225–233.
- [7] Hashorva, E. (2008). A new family of bivariate max-infinitely divisible distributions. *Metrika*, **68**, 289–304.
- [8] Hashorva, E. (2013). Minima and maxima of elliptical arrays and spherical processes. *Bernoulli*, **3**, 886–904.
- [9] Hashorva, E., Kabluchko, Z. and Wübker, A. (2012). Extremes of independent chi-square random vectors. *Extremes*, **15**, 35–42.
- [10] Hashorva, E. and Ling, C. (2016). Maxima of skew elliptical triangular arrays. *Communications in Statistics-Theory and Methods*, **12**, 3692–3705.
- [11] Hashorva, E., Peng, Z. and Weng, Z. (2016). Higher-order expansions of distributions of maxima in a Hüsler-Reiss model. *Methodology and Computing in Applied Probability*, **18**, 181–196.

- [12] Hüsler, J. and Reiss, R.-D. (1989). Maxima of normal random vectors: between independence and complete dependence. *Statistics and Probability Letters*, **7**, 283–286.
- [13] Kabluchko, Z., de Haan, L. and Schlatter, M. (2009). Stationary max-stable fields associated to negative definite functions. *The Annals of Probability*, **37**, 2042–2065.
- [14] Li, T. and Peng, Z. (2016). Moments convergence of powered normal extremes. <http://arxiv.org/abs/1607.02245v1>.
- [15] Liao, X., Peng, Z. (2014). Convergence rate of maxima of bivariate Gaussian arrays to the Hüsler-Reiss distribution. *Statistics and Its Interface*, **7**, 351–362.
- [16] Liao, X., Peng, L., Peng, Z. and Zheng, Y. (2016). Dynamic bivariate normal copula. *Science China Mathematics*, **5**, 955–976.
- [17] Sibuya, M. (1960). Bivariate extreme statistics. *Annals of the Institute of Statistical Mathematics*, **11**, 195–210.
- [18] Zhou, W. and Ling, C. (2016). Higher-order expansions of powered extremes of normal samples. *Statistics and Probability Letters*, **111**, 12–17.